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GROUP THEORETICAL APPROACH TO PATH INTEGRATION ON SPHERES[1]

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The path integral over compact and non-compact spheres (denoted by $\mathcal{H}_\alpha$) is discussed. The short time propagator is decomposed in unitary irreducible representations of the corresponding transformation group G of $\mathcal{H}_\alpha$. Two cases are considered. For $G \simeq \mathcal{H}_\alpha$ the Fourier analysis leads to an expansion in group characters. However, in the general case $\mathcal{H}_\alpha \simeq G/H$ the decomposition gives an expansion in zonal spherical functions of $G \supset H$. The path integral is performed using the orthogonality of the representations. The groups $SO(n)$, $SO(n-1,1)$, $SU(2)$ and $SU(1,1)$ are considered.

1. INTRODUCTION

The path integral on compact and non-compact spheres has recently become very important [1]. A 'sphere' is a space of constant curvature being positive in the compact and negative in the non-compact case. Almost every problem which was exactly solved by path integration was either be mapped onto a path integral on a sphere or on the group manifolds of $SU(2)$ and $SU(1,1)$ which also may be viewed as spaces of constant positive and negative curvature. The purpose of this work is to give a unified treatment of path integrals on spheres using group theoretical methods. We will explicitly discuss the path integral on $SU(2)$, $SU(1,1)$, $S^{n-1} = SO(n)/SO(n-1)$ and $\Lambda^{n-1} \subset SO(n-1,1)/SO(n-1)$.

In order to have a well defined short time formulation of the path integral we embed the spheres into Euclidean or pseudo-Euclidean spaces, respectively.

2. THE FEYNMAN ANSATZ IN PSEUDO-EUCLIDEAN SPACE

The usual n-dimensional Euclidean path integral on the sliced time basis is given by

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{n/2}\prod_{j=1}^{N-1}d^n\mathbf{r}_j, \qquad (2.1)$$

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with short time action

$$S_j = \frac{m}{2\varepsilon}\left[\left(\Delta x_j^1\right)^2 + \cdots + \left(\Delta x_j^n\right)^2\right]. \tag{2.2}$$

The generalization to pseudo-Euclidean space $E_{p,q}$ with metric

$$g_{\mu\nu} = diag\{\underbrace{+1,\ldots,+1}_{p-times},\ \underbrace{-1,\ldots,-1}_{q-times}\} \tag{2.3}$$

leads to the following short time action

$$S_j = \frac{m}{2\varepsilon}\left[\left(\Delta x_j^1\right)^2 + \cdots + \left(\Delta x_j^p\right)^2 - \left(\Delta x_j^{p+1}\right)^2 - \cdots - \left(\Delta x_j^{p+q}\right)^2\right]. \tag{2.4}$$

In order to obtain the correct normalization $\lim_{t_b \to t_a} K(\mathbf{r}_b, \mathbf{r}_a; t_b - t_a) = \delta(\mathbf{r}_b - \mathbf{r}_a)$ the measure in (2.1) is changed to

$$K(\mathbf{r}_b, \mathbf{r}_a; t_b - t_a) = \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left\{\frac{i}{\hbar} S_j\right\} \prod_{j=1}^{N} \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{p/2} \left(\frac{mi}{2\pi\hbar\varepsilon}\right)^{q/2} \prod_{j=1}^{N-1} d^{p+q}\mathbf{r}_j. \tag{2.5}$$

The indefinite metric of $E_{p,q}$

$$(\mathbf{r},\mathbf{r}) = (x^1)^2 + \cdots + (x^p)^2 - (x^{p+1})^2 - \cdots - (x^{p+q})^2 \tag{2.6}$$

has two more consequences. First, in order to have well defined integrals we regularize in the following way:

The integration over compact coordinates $x^1, \ldots, x^p$ is regularized, as usual, by a mass having a small positive imaginary part, $m \to m + i\eta$ ($\eta > 0$), that over the non-compact coordinates $(x^{p+1}, \ldots, x^{p+q})$, however, by a small negative imaginary part of the mass, $m \to m - i\eta$.

Actually this regularization is meaningful in cartesian coordinates. In polar coordinates we will adopt a similar adequate regularization.

The second consequence is that in general we have different subspaces T_α.

$$\begin{aligned} T_{+1} &= \{\mathbf{r} \mid (\mathbf{r},\mathbf{r}) > 0\} \\ T_{-1} &= \{\mathbf{r} \mid (\mathbf{r},\mathbf{r}) < 0\} \\ T_0 &= \{\mathbf{r} \mid (\mathbf{r},\mathbf{r}) = 0\} \end{aligned} \tag{2.7}$$

In this work we will consider quantum mechanics on such subspaces T_α only. For this we introduce generalized polar coordinates r and θ^μ, $\mu = 1, \ldots, p+q-1$:

$$x^\nu = r\, e^\nu(\theta^1, \ldots, \theta^{p+q-1}), \quad \nu = 1, \ldots, p+q. \tag{2.8}$$

The e^ν define unit vectors in T_α, $\mathbf{e} = (e^1, \ldots, e^{p+q})$. The set of all such vectors forms a sphere $\mathcal{H}_\alpha \in T_\alpha$:

$$\mathcal{H}_\alpha = \{\mathbf{e} \mid (\mathbf{e},\mathbf{e}) = \alpha\}, \ \alpha = \{1, -1, 0\}. \tag{2.9}$$

The short time action in T_α now follows to be

$$\begin{aligned} S_j &= \frac{m}{2\varepsilon}(\Delta\mathbf{r}_j, \Delta\mathbf{r}_j) \\ &= \alpha\left(\frac{m}{2\varepsilon}(\Delta r_j)^2 + \frac{m}{\varepsilon} r_j r_{j-1}\right) - \frac{m}{\varepsilon} r_j r_{j-1}(\mathbf{e}_j, \mathbf{e}_{j-1}) \end{aligned} \tag{2.10}$$

The integrals separate into a radial and an angular part, $d^{p+q}\mathbf{r}_j = r_j^{p+q-1} dr_j d\Omega_j$.

3. PATH INTEGRATION IN GENERALIZED POLAR COORDINATES

In this section we develop a general treatment for the path integration on spheres $\mathcal{H}_\alpha$, respectively for the angular path integral in T_α. Our basic idea is as following:

–Choose an appropriate transformation group G of the sphere which acts on $\mathcal{H}_\alpha$ as $\mathbf{a} \xrightarrow{g} \mathbf{e}$, $g \in G$. In other words, each unitvector $\mathbf{e}_j$ may be obtained from a fixed unitvector $\mathbf{a}$ by the transformation $\mathbf{a} \xrightarrow{g_j} \mathbf{e}_j$, $g_j \in G$.

–Now identify the coordinates θ^μ with group parameters of G and change the angular integration $d\Omega_j$ into an integration over the group manifold dg_j .

–As a function on the group manifold the short time propagator may be expanded in unitary irreducible representations $D^\ell_{mn}(g)$ of G.

–Finally the path integration can be performed explicitly by using the orthogonality relations of the representations.

We may consider the two cases $\mathcal{H}_\alpha \simeq G$ where the sphere is isomorphic to the group manifold of G (e.g. $S^3 \simeq SU(2)$) and $\mathcal{H}_\alpha \simeq G/H$ where the sphere is isomorphic to a group quotient (e.g. $S^{n-1} = SO(n)/SO(n-1)$). Here H is the stationary group of $\mathbf{a}$. As will be shown in the first case the Fourier analysis leads to an expansion of the short time propagator in group characters $\chi^{(\ell)}(g) = \mathrm{Tr}(D^\ell(g))$. The only groups where such an isomorphism is possible are $O(2)$, $O(1,1)$, $SU(2)$ and $SU(1,1)$ [2]. In the second case the short time propagator may be expanded in zonal spherical functions $D^\ell_{00}(g)$ which are invariant under transformations of the stationary group H, $D^\ell_{00}(hgh^{-1}) = D^\ell_{00}(g)$, $g \in G$, $h \in H$).

3.1 The Character Expansion

The scalar product $(\mathbf{e}_j, \mathbf{e}_{j-1})$ on the group manifolds of $O(2)$, $O(1,1)$, $SU(2)$ and $SU(1,1)$ may be written as a trace of the spinor representation g of the group

$$(\mathbf{e}_j, \mathbf{e}_{j-1}) = \frac{1}{2}\mathrm{Tr}(g_{j-1}^{-1} g_j). \tag{3.1}$$

For the Abelian groups this is obvious. For $SU(2)$ and $SU(1,1)$ we will explicitly construct the isomorphism. From (3.1) follows that the short time propagator is a function of

$\mathrm{Tr}(g_{j-1}^{-1}g_j)$ and therefore invariant under group transformations $f(g\hat{g}g^{-1}) = f(\hat{g})$. The Fourier expansion on G reduces to an expansion in group characters

$$f(g) = \sum\!\!\!\!\!\!\int_\ell d_\ell\, \chi^{(\ell)}(g)\, \hat{f}(\ell), \qquad \hat{f}(\ell) = \frac{1}{d_\ell}\int_G f(g)\,\chi^{(\ell)*}(g)\, dg. \tag{3.2}$$

In (3.2) d_ℓ is the dimension of the representation for compact groups. If G is non-compact the 'dimension' d_ℓ is given by

$$\int D^\ell_{mn}(g)\; D^{\ell' *}_{mn}(g)\; dg = \frac{\delta(\ell,\ell')}{d_\ell} \tag{3.3}$$

where $\delta(\ell,\ell') = \delta_{\ell\ell'}$ if ℓ is discrete and $\delta(\ell,\ell') = \delta(\ell-\ell')$ if ℓ is continuous. Note that non-compact groups in general have discrete as well as continuous series and that their unitary irreducible representations are infinite-dimensional.

For the character expansion of the short time propagator we obtain

$$K(g_j, g_{j-1};\varepsilon) = \sum\!\!\!\!\!\!\int_\ell K_\ell(r_j, r_{j-1};\varepsilon)\; d_\ell\; \chi^{(\ell)}(g_{j-1}^{-1}g_j). \tag{3.4}$$

With $d\Omega = |\mathcal{H}_\alpha| dg$, where $|\mathcal{H}_\alpha|$ denotes the volume of $\mathcal{H}_\alpha$, we perform the angular integration using the group properties

$$\int_G \chi^{(\ell)}(g_{j-1}^{-1}g_j)\; \chi^{(\ell')}(g_j^{-1}g_{j+1})\; dg_j = \frac{\delta(\ell,\ell')}{d_\ell}\chi^{(\ell)}(g_{j-1}^{-1}g_{j+1}), \qquad \chi^{(\ell)}(g_a^{-1}g_b) = \sum_{mn} D^\ell_{mn}(g_b)\; D^{\ell *}_{mn}(g_a). \tag{3.5}$$

The resulting propagator reads

$$\begin{aligned} K(\mathbf{r}_b, \mathbf{r}_a; t_b - t_a) &= \sum\!\!\!\!\!\!\int_\ell K_\ell(r_b, r_a; t_b - t_a)\; d_\ell\; \chi^\ell(g_a^{-1}g_b) \\ &= \sum\!\!\!\!\!\!\int_\ell K_\ell(r_b, r_a; t_b - t_a) \sum_{mn} Y_{\ell mn}(\mathbf{e}_b)\; Y^*_{\ell mn}(\mathbf{e}_a), \end{aligned} \tag{3.6}$$

where $Y_{\ell mn}(\mathbf{e}) = \sqrt{d_\ell}\; D^\ell_{mn}(g)$. The above representation of the propagator in terms of group characters has already been given by Marinov and Terent'ev [3].

3.2 Expansion in Zonal Spherical Functions

In the general case where $\mathcal{H}_\alpha$ is isomorphic to a group quotient the isomorphism may be established through an $n \times n$ matrix representation $(n = p + q)$ of G. Here $\mathbf{e} = g\mathbf{a}$ and H is stationary group of $\mathbf{a}$, i.e. $h\mathbf{a} = \mathbf{a}, h \in H \subset G$. Obviously the scalar product in the short time action may be expressed as

$$(\mathbf{e}_j, \mathbf{e}_{j-1}) = (g_j\mathbf{a}, g_{j-1}\mathbf{a}) = (g_{j-1}^{-1}g_j\mathbf{a}, \mathbf{a}). \tag{3.7}$$

Again the short time propagator is a function of $g_{j-1}^{-1}g_j$ but now invariant with respect to transformations of H, $f(hgh^{-1}) = f(g)$, $h \in H$. The Fourier expansion simplifies to [4]

$$f(g) = \sum\!\!\!\!\!\!\int_\ell d_\ell\; D^\ell_{00}(g)\, \hat{f}(\ell), \qquad \hat{f}(\ell) = \int_{\mathcal{H}_\alpha} f(g)\; D^{\ell *}_{00}(g)\; d\Gamma. \tag{3.8}$$

$D^\ell_{00}(g)$ are the zonal spherical functions. $d\Gamma$ is the normalized measure on $\mathcal{H}_\alpha$, $d\Gamma = \int_H dg$. Note that $d\Omega = |\mathcal{H}_\alpha| d\Gamma$. For the short time propagator we get

$$K(g_j, g_{j-1};\varepsilon) = \sum\!\!\!\!\!\!\int_\ell K_\ell(r_j, r_{j-1};\varepsilon)\; d_\ell\; D^\ell_{00}(g_{j-1}^{-1}g_j). \tag{3.9}$$

Using

$$\int_{\mathcal{H}_\alpha} D^\ell_{00}(g_{j-1}^{-1}g_j)\; D^{\ell'}_{00}(g_j^{-1}g_{j+1})\; d\Gamma_j = \frac{\delta(\ell,\ell')}{d_\ell} D^\ell_{00}(g_{j-1}^{-1}g_{j+1}) \tag{3.10}$$

the angular integration gives

$$\begin{aligned} K(\mathbf{r}_b, \mathbf{r}_a; t_b - t_a) &= \sum\!\!\!\!\!\!\int_\ell K_\ell(r_b, r_a; t_b - t_a)\; d_\ell D^\ell_{00}(g_a^{-1}g_b) \\ &= \sum\!\!\!\!\!\!\int_\ell K_\ell(r_b, r_a; t_b - t_a)\; \sum_m Y_{\ell m}(\mathbf{e}_b)\; Y^*_{\ell m}(\mathbf{e}_a), \end{aligned} \tag{3.11}$$

where $Y_{\ell m}(\mathbf{e}) = \sqrt{d_\ell}\; D^\ell_{m0}(g)$ are the hyperspherical harmonics on $\mathcal{H}_\alpha$.

4. EXAMPLES

4.1 The Path Integral on $S^{n-1} = SO(n)/SO(n-1)$

The spherical polar coordinates in an n-dimsional Euclidean space E_n are

$$\begin{aligned} x^1 &= r\sin\phi^{(n-1)}\ldots\sin\phi^{(1)}, & 0 &\le r < \infty, \\ x^2 &= r\sin\phi^{(n-1)}\ldots\cos\phi^{(1)}, & 0 &\le \phi^{(1)} < 2\pi, \\ &\;\;\vdots & 0 &\le \phi^{(k)} < \pi\ (k \neq 1). \\ x^n &= r\cos\phi^{(n-1)}, \end{aligned} \tag{4.1}$$

The Feynman-ansatz on E_n is given by

$$K(\mathbf{r}_b, \mathbf{r}_a; t_b - t_a) = \lim_{N\to\infty} \int \prod_{j=1}^N \exp\left\{\frac{i}{\hbar}S_j\right\} \prod_{j=1}^N \left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{n/2} \prod_{j=1}^{N-1} r_j^{n-1} dr_j d\Omega_j, \tag{4.2}$$

with

$$S_j = \frac{m}{2\varepsilon}\Delta r_j^2 + \frac{m}{\varepsilon} r_j r_{j-1}\left[1 - \mathbf{e}_j\cdot\mathbf{e}_{j-1}\right], \tag{4.3}$$

$$d\Omega = \sin^{n-2}\phi^{(n-1)}\ldots\sin^2\phi^{(3)}\sin\phi^{(2)} d\phi^{(n-1)}\ldots d\phi^{(1)}. \tag{4.4}$$

Integrating the invariant volume element dg of $SO(n)$ over all parameters of the stationary group of $\mathbf{a} = (0,\ldots,0,1)$ yields the normalized volume element on S^{n-1}, $d\Gamma = [\Gamma(n/2)/2\pi^{n/2}]\,d\Omega$. Note that $|S^{n-1}| = 2\pi^{n/2}/\Gamma(n/2)$. The unitary irreducible representations labeled by $\ell = 0,1,2...$ have dimension $d_\ell = (2\ell+n-2)(\ell+n-3)!/[\ell!(n-2)!]$. The zonal spherical functions are given by Gegenbauer polynomials of degree ℓ [4]

$$D^\ell_{00}(g) = \frac{(n-3)!\,\ell!}{(\ell+n-3)!}\,C_\ell^{\frac{n-2}{2}}(\cos\Theta), \tag{4.5}$$

where $\cos\Theta = \mathbf{e}\cdot\mathbf{a} = g\mathbf{a}\cdot\mathbf{a}$. The Fourier coefficient $\hat f(\ell)$ of the expansion (3.8) for $f(g) = \exp\{iz(g\mathbf{a}\cdot\mathbf{a})\}$, follows to be $(z = -mr_{j-1}r_j/\hbar\varepsilon)$

$$\hat f(\ell) = \Gamma(n/2)\,(2/z)^{\frac{n-2}{2}}\,i^\ell J_{\ell+\frac{n-2}{2}}(z). \tag{4.6}$$

The expansion

$$\exp\{iz(g\mathbf{a}\cdot\mathbf{a})\} = \sum_{\ell=0}^{\infty} d_\ell\, D^\ell_{00}(g)\,\hat f(\ell) \tag{4.7}$$

is actually the Gegenbauer formula already used for the S^{n-1} path integral [5]. Performing the angular path integral yields

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \sum_{\ell=0}^{\infty} K_\ell(r_a,r_b;t_b-t_a)\,\frac{\Gamma(n/2)}{2\pi^{n/2}}\sum_M Y_{\ell M}(\mathbf{e}_b)Y^*_{\ell M}(\mathbf{e}_a), \tag{4.8}$$

where M stands for the set $(m_1,m_2,\ldots,m_{n-2})$ with $\ell \ge m_1 \ge \ldots \ge m_{n-3} \ge |m_{n-2}| \ge 0$. All m_i are integers. The n-dimensional spherical harmonics $Y_{\ell M}(\mathbf{e})$ are explicitly given in [2]. The remaining radial propagator is

$$\begin{aligned} K_\ell(r_a,r_b;t_b-t_a) &= (r_br_a)^{\frac{1-n}{2}} \lim_{N\to\infty}\int \prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j^\ell\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2}\prod_{j=1}^{N-1}dr_j,\\ S_j^\ell &= \frac{m}{2\varepsilon}\Delta r_j^2 - \left[\left(\ell+\frac{n-2}{2}\right)^2-\frac{1}{4}\right]\frac{\hbar^2\varepsilon}{2mr_jr_{j-1}}, \end{aligned} \tag{4.9}$$

The propagator on S^{n-1}: The propagator on S^{n-1} may be obtained by fixing the radial variables $r_j = R = const.$ via the replacement [5]

$$\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{1/2}\exp\left\{\frac{im}{2\varepsilon\hbar}(\Delta r_j)^2\right\} \Rightarrow \delta(r_j-r_{j-1})\exp\left\{\frac{i\hbar\varepsilon}{8mR^2}(n-1)(n-3)\right\}. \tag{4.10}$$

We find

$$K(\mathbf{e}_b,\mathbf{e}_a;t_b-t_a) = \sum_{\ell=0}^{\infty}\exp\left\{-\frac{i}{\hbar}E_\ell(t_b-t_a)\right\}\frac{\Gamma(n/2)}{2\pi^{n/2}}\sum_M Y_{\ell M}(\mathbf{e}_b)Y^*_{\ell M}(\mathbf{e}_a), \tag{4.11}$$

where $E_\ell = \ell(\ell+n-2)\hbar^2/2mR^2$. The last term in (4.10) which may be understood as a correction term due to the curvature has been chosen such that the ground state $\ell = 0$ has zero energy, $E_0 = 0$. A different choice (e.g. $\exp\{-i\Delta V_c/\hbar\}$ additional to (4.10)) of this unphysical constant gives the same spectrum with $E_0 = \Delta V_c$. This point has already been noticed in ref.[3].

4.2 The Path Integral on $\Lambda^{n-1} \subset SO(n-1,1)/SO(n-1)$

In the n-dimensional pseudo-Euclidean space $E_{n-1,1}$ we take the upper sheet $x^n > 0$ of the two sheeted subspace $T_{-1} = \{\mathbf{r}|(\mathbf{r},\mathbf{r}) < 0\}$. The non-compact sphere in this space, also called pseudosphere, is denoted by Λ^{n-1}. We introduce polar coordinates by

$$\begin{aligned} x^1 &= r\ \sinh\phi^{(n-1)}\sin\phi^{(n-2)}\ldots\sin\phi^{(1)}, \qquad & 0 \le r,\phi^{(n-1)} < \infty,\\ x^2 &= r\ \sinh\phi^{(n-1)}\sin\phi^{(n-2)}\ldots\cos\phi^{(1)}, & 0 \le \phi^{(1)} < 2\pi,\\ &\vdots & 0 \le \phi^{(k)} < \pi\ (k \ne 1, n-1).\\ x^n &= r\ \cosh\phi^{(n-1)}, \end{aligned} \tag{4.12}$$

According to (2.5) the Feynman ansatz reads

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^{\frac{n-1}{2}}\left(\frac{mi}{2\pi\hbar\varepsilon}\right)^{\frac{1}{2}}\prod_{j=1}^{N-1}d^n\mathbf{r}_j, \tag{4.13}$$

with

$$S_j = -\frac{m}{2\varepsilon}\Delta r_j^2 - \frac{m}{\varepsilon}r_jr_{j-1}\left[1+(\mathbf{e}_j,\mathbf{e}_{j-1})\right], \tag{4.14}$$

$$d\Omega = \sinh^{n-2}\phi^{(n-1)}\sin^{n-3}\phi^{(n-2)}\ldots\sin\phi^{(2)}d\phi^{(n-1)}\ldots d\phi^{(1)}. \tag{4.15}$$

As $SO(n-1,1)$ may be viewed as an analytical continuation of $SO(n)$ in the group parameters we may define a normalized measure on Λ^{n-1} by $d\Gamma = [\Gamma(n/2)/2\pi^{n/2}]\,d\Omega$. Besides the parameters we also have to make analytical continuation in the angular momentum ℓ. Actually the fundamental series of unitary irreducible representations D^ℓ of $SO(n-1,1)$ are given by $\ell = -\frac{n-2}{2}+i\rho$, $\rho \in \mathbf{R}$. As D^ℓ and $D^{-\ell-n+2}$ are equivalent we need only $\rho \ge 0$ for a complete set. With (3.3) the dimension of D^ℓ follows to be $d_\ell = 2|\Gamma(\frac{n-2}{2}+i\rho)|^2/[|\Gamma(i\rho)|^2\Gamma(n-1)]$. The zonal spherical functions are given by Gegenbauer functions

$$D^\ell_{00}(g) = \frac{(n-3)!\,\Gamma(\ell+1)}{\Gamma(\ell+n-2)}\,C_\ell^{\frac{n-2}{2}}(\cosh\Theta), \tag{4.16}$$

where $\cosh\Theta = -(\mathbf{e},\mathbf{a}) = -(g\mathbf{a},\mathbf{a})$. Here the expansion (3.8) leads to an integral transformation which is known as generalized Mehler transformation. For $f(g) = \exp\{z(g\mathbf{a},\mathbf{a})\}$ $(z = mr_jr_{j-1}/i\hbar\varepsilon,\ \mathrm{Im}\,m > 0)$ the Fourier coefficient may be calculated [2]

$$\hat f(\ell) = \frac{\Gamma(n/2)}{\sqrt{\pi}}\left(\frac{2}{z}\right)^{(n-2)/2}K_{i\rho}(z). \tag{4.17}$$

Actually the path integral on Λ^{n-1} has been performed[1] for the first time by using this

[1] M. Böhm and G. Junker, Universität Würzburg, March 1986, unpublished. Right after this work we have realized the group theoretical connection leading us to the present treatment which is much more elegant.

integral transformation together with the addition theorem for Gegenbauer functions [6]

$$C^{\nu}_{\ell}(\cosh\Theta) = \frac{\Gamma(2\nu-1)}{[\Gamma(\nu)]^2}\sum_{k=0}^{\infty} 2^{2k}(2k+2\nu-1)\frac{\Gamma(\ell-k+1)[\Gamma(k+\nu)]^2}{\Gamma(\ell+k+2\nu)}\sinh^k\delta\sinh^k\theta \times C^{\nu+k}_{\ell-k}(\cosh\delta)C^{\nu+k}_{\ell-k}(\cosh\theta)C^{\nu-1/2}_{k}(\cos\psi),$$
$$\cosh\Theta = \cosh\delta\cosh\theta - \sinh\delta\sinh\theta\cos\psi. \tag{4.18}$$

Using the asymptotic form of the modified Bessel function $K_{i\rho}(z)$ for large z the path integration results in

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \int_0^{\infty}\frac{\Gamma(n/2)}{2\pi^{n/2}}\,d_\ell\, D^{\ell}_{00}(g_a^{-1}g_b)\,K_\rho(r_b,r_a,t_b-t_a)d\rho, \tag{4.19}$$

with

$$K_\rho(r_b,r_a,t_b-t_a) = (r_br_a)^{\frac{1-n}{2}}\lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S^{\rho}_j\right\}\prod_{j=1}^{N}\left(\frac{mi}{2\pi\hbar\varepsilon}\right)^{1/2}\prod_{j=1}^{N-1}dr_j,$$
$$S^{\rho}_j = -\frac{m}{2\varepsilon}\Delta r_j^2 - \frac{\rho^2+1/4}{2mr_jr_{j-1}}\hbar^2\varepsilon. \tag{4.20}$$

The propagator on Λ^{n-1}: The propagator on Λ^{n-1} may be obtained analogous to S^{n-1} by replacing

$$\sqrt{\frac{mi}{2\pi\hbar\varepsilon}}\exp\left\{\frac{-im\Delta r_j^2}{2\hbar\varepsilon}\right\} \Rightarrow \delta(r_j-r_{j-1})\exp\left\{-\frac{i\hbar\varepsilon}{8mR^2}(n-1)(n-3)\right\} \tag{4.21}$$

The propagator on the n-dimensional pseudosphere Λ^{n-1} is

$$K(\mathbf{e}_b,\mathbf{e}_a;t_b-t_a) = \int_0^{\infty}d\rho\exp\left\{-\frac{i}{\hbar}E_\rho(t_b-t_a)\right\}\frac{\Gamma(n/2)}{2\pi^{n/2}}\sum_{k,M}Y_{\rho kM}(\mathbf{e}_b)Y^{*}_{\rho kM}(\mathbf{e}_a), \tag{4.22}$$

with the continuous spectrum $E_\rho = \{\rho^2+(n-2)^2/4\}\hbar^2/2mR^2$. Up to an unimportant constant this is the spectrum of a free particle with momentum $p=\rho\hbar/R$. The hyperspherical harmonics $Y_{\rho kM}(\mathbf{e})$ may be expressed in terms of the harmonics $Y_{kM}(\mathbf{e}^{n-1})$ of the maximal compact subsphere $S^{n-2}\subset\Lambda^{n-1}$. $Y_{\rho kM}(\mathbf{e}) = Z_{\rho k}(\phi^{(n-1)})Y_{kM}(\mathbf{e}^{n-1})$ where

$$Z_{\rho k}(\phi) = \frac{\Gamma(\frac{n-2}{2}+k+i\rho)}{\Gamma(i\rho)}\sinh^{\frac{3-n}{2}}\phi\, P^{\frac{3-n}{2}-k}_{-\frac{1}{2}+i\rho}(\cosh\phi). \tag{4.23}$$

Note that the first part of equ.(3.11) provides in this case an integral representation of the propagator in closed form

$$K(\mathbf{e}_b,\mathbf{e}_a;t_b-t_a) = \frac{\Gamma(n/2)(n-3)!}{2\pi^{n/2}}\int_0^{\infty}d\rho\; d_\ell\exp\left\{-\frac{i}{\hbar}E_\rho(t_b-t_a)\right\} \times\frac{\Gamma(\ell+1)}{\Gamma(\ell+n-2)}C^{(n-2)/2}_{\ell}(\cosh\Theta_{ab}) \tag{4.24}$$

with $\cosh\Theta_{ab} = -(\mathbf{e}_b,\mathbf{e}_a)$ and $\ell=(2-n)/2+i\rho$.

4.3 Path Integration over the $SU(2)$ Manifold

The isomorphism of $SU(2)$ with the sphere in E_4

$$\begin{aligned} x^1 &= r\sin\theta/2\sin(\varphi-\psi)/2, & 0&\le r<\infty,\\ x^2 &= r\sin\theta/2\cos(\varphi-\psi)/2, & 0&\le\varphi<2\pi,\\ x^3 &= r\cos\theta/2\sin(\varphi+\psi)/2, & 0&\le\theta<\pi,\\ x^4 &= r\cos\theta/2\cos(\varphi+\psi)/2, & 0&\le\psi<4\pi \end{aligned} \tag{4.25}$$

may be established by the spinor representation of $SU(2)$

$$g = \begin{pmatrix} e^4+ie^3 & ie^1+e^2\\ ie^1-e^2 & e^4-ie^3\end{pmatrix}. \tag{4.26}$$

Note that $e^i = x^i/r$. The polar coordinates can be identified with the usual Euler angles of $SU(2)$. As $|S^3| = 2\pi^2$ we find $d\Omega = 2\pi^2dg$ and the Feynman ansatz reads

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}\exp\left\{\frac{i}{\hbar}S_j\right\}\prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\varepsilon}\right)^2\prod_{j=1}^{N-1}2\pi^2r_j^3dr_j\,dg_j,$$
$$S_j = \frac{m}{2\varepsilon}\Delta r_j^2 + \frac{m}{\varepsilon}r_jr_{j-1}\left[1-\frac{1}{2}\mathrm{Tr}(g_{j-1}^{-1}g_j)\right]. \tag{4.27}$$

The character expansion for $SU(2)$ gives [5] ($2z = mr_jr_{j-1}/i\varepsilon\hbar$)

$$\exp\{z\,\mathrm{Tr}(g)\} = \sum_\ell d_\ell\,\frac{1}{z}\,I_{2\ell+1}(2z)\,\chi^{(\ell)}(g), \tag{4.28}$$

where $\chi^{(\ell)}(g) = \sin[(\ell+\frac{1}{2})\Theta]/\sin(\Theta/2)$, $\cos(\Theta/2) = \cos(\theta/2)\cos[(\varphi+\psi)/2]$, $d_\ell = 2\ell+1$ and $\ell = 0,\frac{1}{2},1,\frac{3}{2},\ldots$. Performing the angular integration yields

$$K(\mathbf{r}_b,\mathbf{r}_a;t_b-t_a) = \sum_\ell\frac{d_\ell}{2\pi^2}K_\ell(r_b,r_a;t_b-t_a)\,\chi^{(\ell)}(g_a^{-1}g_b). \tag{4.29}$$

The radial path integral is given by (4.9) with $n=4$.

The $SU(2)$ propagator: Besides the replacement (4.10) for $n=4$ we change the normalization $2\pi^2dg$ to dg and obtain the $SU(2)$ propagator

$$K(g_b,g_a;t_b-t_a) = \sum_\ell d_\ell\exp\left\{-\frac{i}{\hbar}E_\ell(t_b-t_a)\right\}\chi^{(\ell)}(g_a^{-1}g_b),$$
$$E_\ell = \frac{2\hbar^2}{mR^2}\ell(\ell+1). \tag{4.30}$$

4.4 Path Integration over the $SU(1,1)$ Manifold

The $SU(1,1)$ manifold is isomorphic to the sphere $\mathcal{H}_{+1}\in E_{2,2}$ with metric $(\mathbf{e},\mathbf{e}) =$

$-(e^1)^2 - (e^2)^2 + (e^3)^2 + (e^4)^2 = +1$. The parametrization is analogous to (4.25)

$$\begin{aligned} x^1 &= r\sinh\theta/2\,\sin(\psi-\varphi)/2, & 0 &\le r < \infty,\\ x^2 &= r\sinh\theta/2\,\cos(\psi-\varphi)/2, & 0 &\le \varphi < 2\pi,\\ x^3 &= r\cosh\theta/2\,\sin(\psi+\varphi)/2, & 0 &\le \theta < \infty,\\ x^4 &= r\cosh\theta/2\,\cos(\psi+\varphi)/2, & 0 &\le \psi < 4\pi. \end{aligned} \tag{4.31}$$

The isomorphism is explicitly given by

$$g = \begin{pmatrix} e^4 + ie^3 & e^1 - ie^2 \\ e^1 + ie^2 & e^4 - ie^3 \end{pmatrix}. \tag{4.32}$$

For more details we refer to [2]. Obviously the spinor representation (4.32) of $SU(1,1)$ is found by analytical continuation in the Euler angle $\theta \to -i\theta$ of $SU(2)$. Note that the invariant volume element is $dg = 2\pi^2 d\Omega$. The angular momentum becomes complex, too. The two fundamental series of $SU(1,1)$ are [7]

$$\ell = -\tfrac{1}{2} + i\rho \begin{cases} \rho \ge 0, m = 0, \pm 1, \pm 2, \dots & for\ \sigma = 0 \\ \rho > 0, m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \dots & for\ \sigma = \tfrac{1}{2} \end{cases} \tag{4.33}$$

$$\ell = -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1, \dots \begin{cases} m = \ell+1, \ell+2, \dots & for\ \sigma = + \\ m = -\ell-1, -\ell-2, \dots & for\ \sigma = - \end{cases} \tag{4.34}$$

and the dimension of the unitary irreducible representation D^ℓ is

$$d_\ell = \begin{cases} 2\ell+1 & for\ \ell \ge 0, \quad \delta(\ell,\ell') = \delta_{\ell\ell'}, \\ 2\rho\tanh\pi(\rho + i\sigma) & for\ \ell = -1/2 + i\rho, \quad \delta(\ell,\ell') = \delta(\rho-\rho'). \end{cases} \tag{4.35}$$

A complete set in the Hilbert space is given by $\ell \ge 0$ ($\sigma = +$ or $-$) and $\ell = -\frac{1}{2} + i\rho$ ($\sigma =$ 0 and $\frac{1}{2}$). The corresponding Feynman ansatz in $E_{2,2}$ is

$$\begin{aligned} K(\mathbf{r}_b, \mathbf{r}_a; t_b - t_a) &= \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left\{\frac{i}{\hbar} S_j\right\} \prod_{j=1}^{N} \left(\frac{m}{2\pi i\hbar\epsilon}\right)\left(\frac{im}{2\pi\hbar\epsilon}\right) \prod_{j=1}^{N-1} r_j^3 dr_j\, 2\pi^2 dg_j, \\ S_j &= \frac{m}{2\epsilon}\Delta r_j^2 + \frac{m}{\epsilon} r_j r_{j-1}\left[1 - \frac{1}{2}\mathrm{Tr}(g_{j-1}^{-1} g_j)\right]. \end{aligned} \tag{4.36}$$

For $f(g) = \exp\{-(iz/2)\mathrm{Tr}(g)\}$ with $z = (mr_j r_{j-1}/\hbar\epsilon)$ the Fourier coefficient of the character expansion is for large z [8]

$$\hat{f}(\ell) \simeq \frac{1}{2\pi^2}\frac{2\pi}{iz}\left(\frac{2\pi i}{z}\right)^{1/2} \exp\left\{-iz - i\frac{(2\ell+1)^2 - 1/4}{2z}\right\}. \tag{4.37}$$

Performing the angular integration gives

$$K(\mathbf{r}_b, \mathbf{r}_a; t_b - t_a) = \sum_\sigma \sum\!\!\!\!\!\!\int_\ell \frac{d_\ell}{2\pi^2} \chi^{(\ell)}(g_a^{-1} g_b) K_\ell(r_b, r_a; t_b - t_a) \tag{4.38}$$

with the remaining radial path integral

$$\begin{aligned} K_\ell(r_b, r_a; t_b - t_a) &= (r_b r_a)^{-3/2} \lim_{N\to\infty} \int \prod_{j=1}^{N} \exp\left\{\frac{i}{\hbar} S_j^\ell\right\} \prod_{j=1}^{N}\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{1/2} \prod_{j=1}^{N-1} dr_j, \\ S_j^\ell &= \frac{m}{2\epsilon}\Delta r_j^2 - \left[(2\ell+1)^2 - 1/4\right]\frac{\hbar^2\epsilon}{2mr_j r_{j-1}}. \end{aligned} \tag{4.39}$$

The $SU(1,1)$ propagator: Proceeding in the same way as for $SU(2)$ we find the propagator on the $SU(1,1)$ group manifold

$$\begin{aligned} K(g_b, g_a; t_b - t_a) &= \sum_\sigma \sum\!\!\!\!\!\!\int_\ell d_\ell \exp\left\{-\frac{i}{\hbar} E_\ell (t_b - t_a)\right\} \chi^{(\ell)}(g_a^{-1} g_b), \\ E_\ell &= \begin{cases} \dfrac{2\hbar^2}{mR^2}\ell(\ell+1) & for\ \ell \ge 0, \\ -\dfrac{2\hbar^2}{mR^2}(\rho^2 + 1/4) & for\ \ell = -\tfrac{1}{2} + i\rho. \end{cases} \end{aligned} \tag{4.40}$$

5. DISCUSSION

In this work we have presented a unified treatment of Feynman path integrals in generalized polar coordinates of compact and non-compact spheres. Two group theoretical methods have been developed. The first, the character expansion, is applicable to the path integral on the group manifolds of $O(2)$, $O(1,1)$, $SU(2)$ and $SU(1,1)$. The general treatment, the Fourier decomposition in zonal spherical functions, gives in these cases the same results. Combination of both methods leads to integrable path integrals in bispherical polar coordinates of E_{p+q} and $E_{p,q}$ [2].

For curved manifolds usually a correction has to be taken into account which is propotional to the curvature. It has become apparent at this conference that there are still disagreements in the present literature about the explicit form of this correction. As Schulman mentioned in his talk we have to wait for an experiment to solve this problem [9]. However, we are dealing with spaces of constant curvature called spheres. Therefore the correction is an additional overall constant appearing in the energy spectrum. Obviously such a correction is physically unimportant (i.e. it can be chosen in any way) and recent critisism [10] about this point is not justified.

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